Last Name:

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Practice Problems: Exercise 5 – Microengineering 110

Spring 2025

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1. A new battery's voltage may be acceptable (A) or unacceptable (U). A certain flashlight requires two batteries, so batteries will be independently selected and tested until two acceptable ones have been found. Suppose that 90% of all batteries have acceptable voltages. Let Y denote the number of batteries that must be tested.

    a. What is p(2), that is, P(Y = 2)?

p(2) = P(Y = 2) = P(first 2 batteries are acceptable) = P(AA) = (.9)(.9) = .81.

    b. What is p(3)? [Hint: There are two different outcomes that result in Y = 3.

p(3) = P(Y = 3) = P(UAA or AUA) = (.1)(.9)2 + (.1)(.9)2 = 2[(.1)(.9)2] = .162.

    c. To have Y = 5, what must be true of the fifth battery selected? List the four outcomes for which Y = 5 and then determine p(5).

The fifth battery must be an A, and exactly one of the first four must also be an A. Thus, p(5) = P(AUUUA or UAUUA or UUAUA or UUUAA) = 4[(.1)3(.9)2] = .00324.

    d. Use the pattern in your answers for parts (a)–(c) to obtain a general formula for p(y).

p(y) = P(the yth is an A and so is exactly one of the first y – 1) = (y – 1)(.1)y–2(.9)2, for y = 2, 3, 4, 5, ….

2. Two fair six-sided dice are tossed independently. Let M = the maximum of the two tosses (so M(1,5) = 5, M(3,3) = 3, etc.).

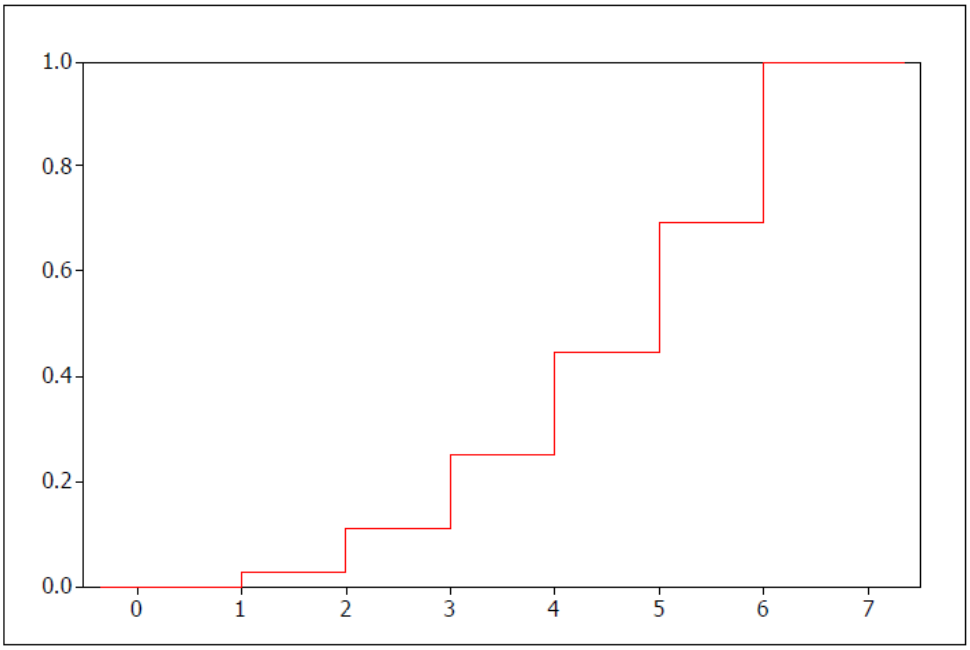
    a. What is the pmf of M? [Hint: First determine p(1), then p(2), and so on.]

$p(1) = P(M = 1) = P(\{(1,1)\}) = \frac{1}{36}$; $p(2) = P(M = 2) = P(\{(1,2)(2,1)(2,2)\}) = \frac{3}{36}$; $p(3) = P(M = 3) = P(\{(1,3)(2,3)(3,1)(3,2)(3,3)\}) = \frac{5}{36}$. Continuing the pattern, $p(4) = \frac{7}{36}$, $p(5) = \frac{9}{36}$, and $p(6) = \frac{11}{36}$.

    b. Determine the cdf of M and graph it.

Using the values in **a**,

$$F(m) = \begin{cases} 0 & m < 1 \\ \frac{1}{36} & 1 \le m < 2 \\ \frac{4}{36} & 2 \le m < 3 \\ \frac{9}{36} & 3 \le m < 4 \\ \frac{16}{36} & 4 \le m < 5 \\ \frac{25}{36} & 5 \le m < 6 \\ 1 & m \ge 6 \end{cases}$$



3. The pmf of the amount of memory X (GB) in a purchased flash drive was given as

x	1	2	4	8	16
$p(x)$	.05	.10	.35	.40	.10

Compute the following:

a. E(X)

$$E(X) = \sum_{\text{all } x} xp(x) = 1(.05) + 2(.10) + 4(.35) + 8(.40) + 16(.10) = 6.45 \text{ GB.}$$

b. V(X)

$$V(X) = \sum_{\text{all } x} (x - \mu)^2 p(x) = (1 - 6.45)^2(.05) + (2 - 6.45)^2(.10) + \ldots + (16 - 6.45)^2(.10) = 15.6475.$$

c. The standard deviation of X

$$\sigma = \sqrt{V(X)} = \sqrt{15.6475} = 3.956 \text{ GB}.$$

4. A company that produces fine glassware knows from experience that 10% of its goblets have cosmetic flaws and must be classified as "seconds."

a. Among six randomly selected goblets, how likely is it that only one is a second?

Let X be the number of "seconds," so $X \sim \text{Bin}(6, .10)$.

$$P(X = 1) = \binom{n}{x} p^x (1-p)^{n-x} = \binom{6}{1} (.1)^1 (.9)^5 = .3543.$$

b. Among six randomly selected goblets, what is the probability that at least two are seconds?

$$P(X \ge 2) = 1 - [P(X = 0) + P(X = 1)] = 1 - \left[\binom{6}{0}(.1)^0(.9)^6 + \binom{6}{1}(.1)^1(.9)^5\right] = 1 - [.5314 + .3543] = .1143.$$

c. If goblets are examined one by one, what is the probability that at most five must be selected to find four that are not seconds?

Either 4 or 5 goblets must be selected.

Select 4 goblets with zero defects: $P(X = 0) = \binom{4}{0}(.1)^0(.9)^4 = .6561$.

Select 4 goblets, one of which has a defect, and the 5$^{\text{th}}$ is good: $\left[\binom{4}{1}(.1)^1(.9)^3\right] \times .9 = .26244$

So, the desired probability is $.6561 + .26244 = .91854$.

5. Suppose that the number of drivers who travel between a particular origin and destination during a designated time period has a Poisson distribution with parameter $\mu = 20$. What is the probability that the number of drivers will

a. Be at most 10?

Let $X \sim \text{Poisson}(\mu = 20)$.

$$P(X \le 10) = F(10; 20) = .011.$$

b. Exceed 20?

$$P(X > 20) = 1 - F(20; 20) = 1 - .559 = .441.$$

c. Be between 10 and 20, inclusive?

$$P(10 \le X \le 20) = F(20; 20) - F(9; 20) = .559 - .005 = .554;$$

d. Be strictly between 10 and 20?

$$P(10 < X < 20) = F(19; 20) - F(10; 20) = .470 - .011 = .459.$$

e. Be within 2 standard deviations of the mean value?

$E(X) = \mu = 20$, so $\sigma = \sqrt{20} = 4.472$. Therefore, $P(\mu - 2\sigma < X < \mu + 2\sigma) =$
$P(20 - 8.944 < X < 20 + 8.944) = P(11.056 < X < 28.944) = P(X \le 28) - P(X \le 11) =$
$F(28; 20) - F(11; 20) = .966 - .021 = .945.$

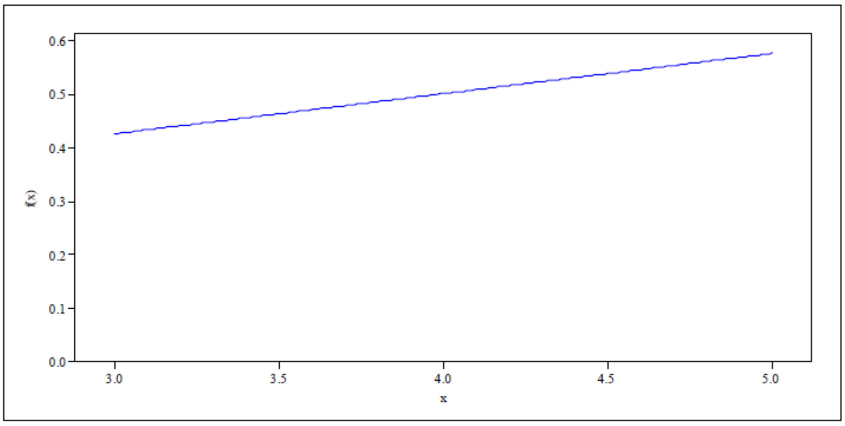
6. The current in a certain circuit as measured by an ammeter is a continuous random variable X with the following density function:

$$f(x) = \begin{cases} .075x + .2 & 3 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$$

a. Graph the pdf and verify that the total area under the density curve is indeed 1.

The pdf is the straight-line function graphed below on [3, 5]. The function is clearly non-negative; to verify its integral equals 1, compute:

$$\int_3^5 (.075x + .2)\,dx = .0375x^2 + .2x\Big]_3^5 = (.0375(5)^2 + .2(5)) - (.0375(3)^2 + .2(3))$$
$$= 1.9375 - .9375 = 1$$



b. Calculate P(X ≤ 4). How does this probability compare to P(X < 4)?

$P(X \le 4) = \int_3^4 (.075x + .2)\,dx = .0375x^2 + .2x\Big]_3^4 = (.0375(4)^2 + .2(4)) - (.0375(3)^2 + .2(3))$
$= 1.4 - .9375 = .4625$. Since X is a continuous rv, $P(X < 4) = P(X \le 4) = .4625$ as well.

c. Calculate P(3.5 ≤ X ≤ 4.5) and also P(4.5 < X).

$$P(3.5 \le X \le 4.5) = \int_{3.5}^{4.5} (.075x + .2)\,dx = .0375x^2 + .2x\Big]_{3.5}^{4.5} = \cdots = .5\,.$$

$$P(4.5 < X) = P(4.5 \le X) = \int_{4.5}^{5} (.075x + .2)\,dx = .0375x^2 + .2x\Big]_{4.5}^{5} = \cdots = .278125\,.$$

7. Spray drift is a constant concern for pesticide applicators and agricultural producers. The inverse relationship between droplet size and drift potential is well known. The normal distribution with mean 1050 μm and standard deviation 150 μm was a reasonable model for droplet size for water (the "control treatment") sprayed through a 760 ml/min nozzle.

   a. What is the probability that the size of a single droplet is less than 1500 μm? At least 1000 μm?

$P(X < 1500) = P(Z < 3) = \Phi(3) = .9987$; $P(X \geq 1000) = P(Z \geq -.33) = 1 - \Phi(-.33) = 1 - .3707 = .6293$.

   b. What is the probability that the size of a single droplet is between 1000 and 1500 μm?

$$P(1000 < X < 1500) = P(-.33 < Z < 3) = \Phi(3) - \Phi(-.33) = .9987 - .3707 = .6280$$

   c. If the sizes of five independently selected droplets are measured, what is the probability that exactly two of them exceed 1500 μm?

Let Y = the number of droplets, out of 5, that exceed 1500 μm. Then Y is binomial, with $n = 5$ and $p = .0013$ from **a**. So, $P(Y = 2) = \binom{5}{2}(.0013)^2(.9987)^3 \approx 1.68 \times 10^{-5}$.